Approximation by trigonometric polynomials in weighted Morrey spaces

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Abstract

In this paper we investigate the best approximation by trigonometric polynomials in weighted Morrey spaces $\mathcal{M}_{p,\lambda}(I_0,w)$, where the weight function w is in the Muckenhoupt class $A_p(I_0)$ with $1<p<\infty$ and $I_0=[0,2\pi]$. We prove the direct and inverse theorems of approximation by trigonometric polynomials in the spaces $\widetilde{\mathcal{M}}_{p,\lambda}(I_0,w)$ the closure of $C^\infty(I_0)$ in $\mathcal{M}_{p,\lambda}(I_0,w)$. We give the characterization of $K-$functionals in terms of the modulus of smoothness and obtain the Bernstein type inequality for trigonometric polynomials in the spaces $\mathcal{M}_{p,\lambda}(I_0,w)$.

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1 Introduction and main results

Morrey spaces date back to the paper [21] published in 1938, where C.B. Morrey studied the local behavior of solutions to elliptic differential equations. Now Morrey spaces are used in several branches of mathematics, first of all in analysis, PDE and potential theory.

Let $0\le\lambda\le 1$ and $1\le p<\infty$. For intervals in this paper we write

$$I_0=[0,2\pi],\quad I(x,r)=(x-r,x+r)\subset\mathbb{R},\quad I_0(x,r)=I(x,r)\cap I_0.$$

By w we always denote a weight such that a positive, $2\pi-$periodic and locally integrable function on I_0. The weighted Morrey space $\mathcal{M}_{p,\lambda}(I_0,w)$ is defined as the set of all functions $f\in L_p(I_0,w)$ such that

$$\|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)}=\sup_{\substack{x\in I_0\\ 0<r<2\pi}} w(I(x,r))^{-\frac{\lambda}{p}}\|f\|_{L_{p,w}(I_0(x,r))}<\infty,$$

where

$$\|f\|_{L_{p,w}(I_0(x,r))}=\Big(\int_{I_0(x,r)}|f(t)|^p\,w(t)dt\Big)^{1/p}.$$

Under this definition $\mathcal{M}_{p,\lambda}(I_0,w)$ is a Banach space. If $w=1$ and $0<\lambda<1$, then $\mathcal{M}_{p,\lambda}(I_0,w)=\mathcal{M}_{p,\lambda}(I_0)$. If $\lambda=0$, we get the weighted Lebesgue spaces $L_{p,w}(I_0)$. If $\lambda=1$, $\mathcal{M}_{p,1}(I_0,w)=L_{\infty,w}(I_0)$ by the Lebesgue differentiation theorem with respect to w (see [28]). Denote by $C^\infty(I_0)$ the set of all functions that are realized as the restriction to I_0 of elements in $C^\infty(\mathbb{R})$. The weighted Morrey space $\mathcal{M}_{p,\lambda}(I_0,w)$ does not have $C^\infty(I_0)$ as a dense closed subspace; the case $w=1$ was proved in [27]. We define $\widetilde{\mathcal{M}}_{p,\lambda}(I_0,w)$ as the closure of $C^\infty(I_0)$ in $\mathcal{M}_{p,\lambda}(I_0,w)$.

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Definition 1.1. (Muckenhoupt classes) A weight function w is in the Muckenhoupt class $A_p(I_0)$ with $1 < p < \infty$ if there exists $C > 1$ such that for any interval $I \subset I_0$

$$\left(\frac{1}{|I|}\int_I w(x)dx\right)\left(\frac{1}{|I|}\int_I w(x)^{1-p'}dx\right)^{p-1} \leq C, \tag{1.1}$$

where $1/p + 1/p' = 1$ and the infimum of C satisfying the inequality (1.1) is denoted by $[w]_{A_p(I_0)}$. We define $A_\infty(I_0) = \bigcup_{1<p<\infty} A_p(I_0)$.

Hardy-Littlewood maximal function Mf of f on I_0 is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|I(x,r)|}\int_{I_0(x,r)} |f(t)|dt, \ x \in I_0.$$

When $p = 1$, $w \in A_1(I_0)$ if there exists $C > 1$ such that for almost every $x \in I_0$,

$$Mw(x) \leq Cw(x), \tag{1.2}$$

and the infimum of C satisfying the inequality (1.2) is denoted by $[w]_{A_1}$.

We will need the following theorem on the boundedness of M in the spaces $\mathcal{M}_{p,\lambda}(I_0, w)$ which proved by Y. Komori, S. Shirai [16].

Theorem A. *Let $1 < p < \infty$, $0 \leq \lambda < 1$ and $w \in A_p(I_0)$. Then the Hardy-Littlewood maximal operator M is bounded on $\mathcal{M}_{p,\lambda}(I_0, w)$.*

The fundamental problem in approximation theory consists in finding for a complicated function from a normed space, a simple function (polynomial or rational function) to approximate. We denote by $\mathcal{P}_n$ as the set of trigonometric polynomials having degree not exceeding n and $C(I_0)$ the set of $2\pi-$periodic continuous functions. Let $f \in C(I_0)$ and $E_n(f)$ be the best approximation of f by the trigonometric polynomials, i.e.,

$$E_n(f)_{C(I_0)} = \inf_{T_n \in \mathcal{P}_n} \left\|f - T_n\right\|_{C(I_0)}.$$

The Weierstrass well-known theorem on the approximation of the continuous function by the trigonometric polynomials and its quantitative refinement represented by Jackson inequality

$$E_n(f)_{C(I_0)} \leq C\omega\Big(f, \frac{1}{n}\Big)$$

are one of the basics of the Approximation Theory, where $\omega(f, \delta)$, $\delta > 0$ is the modulus of continuity of f (see [5]). The analog of Jackson inequality is correct for the mean approximation and higher order modulus of continuity as well (see [25]). S. Bernstein [1] obtained the reversed estimations of Jackson's inequality in the space of continuous functions for some specific cases. Later E.S. Quade [23], S.B. Stechkin [24], A.F. Timan [25], A.F. and M.F. Timan [26] etc. proved the reversed type inequalities of Jackson's inequality, including $L_p, 1 < p < \infty$, spaces. These type inequalities played an important role in the investigation of properties of the conjugate functions, in the study of absolute convergent Fourier series [24], and in the related problems. For the approximation in weighted and nonweighted Lebesgue spaces, weighted Lorentz spaces, and Smirnov-Orlicz spaces the sufficiently wide presentation can be found in the works [7], [12], [13], [14], [15], [17] and [18]. In [2] the authors investigate the best approximation by trigonometric polynomials in Morrey space $L_{p,\lambda}(I_0)$.

Let $\mathbb{N} := \{1, 2, ...\}$ and

$$\triangle_t^r f(x) := \sum_{s=0}^{r} \binom{r}{s} (-1)^{r+s+1} f(x+st), \ r \in \mathbb{N}$$

for $f \in \mathcal{M}_{p,\lambda}(I_0, w)$, $\quad 0 \leq \lambda \leq 1$, $p \geq 1$ and

$$\sigma_\delta^r(f)(x) := \frac{1}{\delta} \int_0^\delta |\triangle_t^r f(x)| dt. \tag{1.3}$$

One of the main problems observed in the investigations on the approximation theory is the correct definition of the modulus of smoothness that will provide us with a better tool to deal with the rate of the best approximation, inverse theorems and also some other similar problems. Now we define the modulus of smoothness in $\mathcal{M}_{p,\lambda}(I_0, w)$.

Definition 1.2. Let $g \in \mathcal{M}_{p,\lambda}(I_0, w)$, $0 \leq \lambda \leq 1$, $1 < p < \infty$ and $h > 0$. Then the function $\Omega^r(g, \cdot, \mathcal{M}_{p,\lambda}(I_0, w)) : [0, \infty] \to [0, +\infty)$, defined by

$$\Omega^r(g, h, \mathcal{M}_{p,\lambda}(I_0, w)) := \sup_{0<\delta\leq \min\{2\pi, h\}} \|\sigma_\delta^r(g)\|_{\mathcal{M}_{p,\lambda}(I_0,w)}, \quad r \in \mathbb{N}$$

is called the rth modulus of smoothness of g in $\mathcal{M}_{p,\lambda}(I_0, w)$.

From Corollary 5 we get $\Omega^r(g, h, \mathcal{M}_{p,\lambda}(I_0, w)) \leq c\|g\|_{\mathcal{M}_{p,\lambda}(I_0,w)}$ for every $g \in \mathcal{M}_{p,\lambda}(I_0, w)$, and

$$\Omega^r(g_1 + g_2, \cdot, \mathcal{M}_{p,\lambda}(I_0, w)) \leq \Omega^r(g_1, \cdot, \mathcal{M}_{p,\lambda}(I_{0,w})) + \Omega^r(g_2, \cdot, \mathcal{M}_{p,\lambda}(I_0, w))$$

for $g_1, g_2 \in \mathcal{M}_{p,\lambda}(I_0, w)$, where $0 \leq \lambda \leq 1$, $1 < p < \infty$.

For $f \in \mathcal{M}_{p,\lambda}(I_0, w)$, $0 \leq \lambda \leq 1$ and $p \geq 1$, we denote

$$E_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} := \inf_{T_n \in \mathcal{P}_n} \|f - T_n\|_{\mathcal{M}_{p,\lambda}(I_0,w)}$$

the minimal error of approximation of f in the class $\mathcal{P}_n$.

The homogeneous Sobolev-Morrey space $\dot{W}^r\mathcal{M}_{p,\lambda}(I_0, w)$ is defined as the set of all functions $f \in L_1^{loc}(I_0)$ for which the weak derivative $f^{(r)}$ exists on I_0 and

$$\|f\|_{\dot{W}^r\mathcal{M}_{p,\lambda}(I_0,w)} = \|f^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} < \infty.$$

The non-homogeneous Sobolev–Morrey space $W^r\mathcal{M}_{p,\lambda}(I_0, w)$ is the subset of $\dot{W}^r\mathcal{M}_{p,\lambda}(I_0, w)$, consisting of all functions $f \in \dot{W}^r\mathcal{M}_{p,\lambda}(I_0, w)$ for which

$$\|f\|_{W^r\mathcal{M}_{p,\lambda}(I_0,w)} := \|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \|f^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} < \infty.$$

Note that if $\lambda = 0$, then $W^r_{p,w}(I_0) \equiv W^r\mathcal{M}_{p,0}(I_0, w)$ is weighted Sobolev space and if $w = 1$, then $W^r_{p,\lambda}(I_0) \equiv W^r\mathcal{M}_{p,\lambda}(I_0, 1)$ is the Sobolev-Morrey space.

For $f \in \mathcal{M}_{p,\lambda}(I_0, w)$, $0 \le \lambda \le 1$, $1 < p < \infty$, $w \in A_p(I_0)$ and $r \ge 1$ the K- functional is defined as follows

$$K_r(f, t)_{\mathcal{M}_{p,\lambda}(I_0,w)} = \inf_{g\in W^r\mathcal{M}_{p,\lambda}(I_0,w)} \{\|f - g\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + t^r\|g^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\}, t > 0.$$

In this paper we study the direct and inverse problems of approximation theory in weighted Morrey spaces $\widetilde{\mathcal{M}}_{p,\lambda}(I_0, w)$, the closure of the set of trigonometric polynomials in $\mathcal{M}_{p,\lambda}(I_0, w)$ with $1 < p < \infty$ and $w \in A_p(I_0)$. We give a characterization of K-functionals in terms of the modulus of smoothness in weighted Morrey spaces $\mathcal{M}_{p,\lambda}(I_0, w)$.

The direct result can be formulated as follows:

Theorem 1.3. Let $f \in \widetilde{\mathcal{M}}_{p,\lambda}(I_0, w)$, $0 \le \lambda \le 1$, $1 < p < \infty$ and $w \in A_p(I_0)$. Then for every $r \in \mathbb{N}$ we have

$$E_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} \le C\ \Omega^r \left(f, \frac{1}{n}, \mathcal{M}_{p,\lambda}(I_0, w)\right), \quad n \ge r$$

with a constant $C > 0$ independent of f and n.

Similar result in Lebesgue spaces $L_p(I_0)$, in terms of usually modulus of smoothness, defined as

$$\sup_{|t|\le h} \|\triangle^r_t f(x)\|_{L_p(I_0)}, \quad h > 0,\ r \in \mathbb{N}$$

was proved by S.B. Stechkin in [24]. In weighted Lebesgue spaces $L_p(I_0, w) \equiv L_{p,w}(I_0)$ when the weight w is in the Muckenhoupt class $A_p(I_0)$, similar result was proved by N.X. Ky in [17].

The inverse result can be formulated as follows.

Theorem 1.4. Let $f \in \widetilde{\mathcal{M}}_{p,\lambda}(I_0, w)$, $0 \le \lambda < 1$, $1 < p < \infty$ and $w \in A_p(I_0)$. Then for every $r \in \mathbb{N}$ we have

$$\Omega^r\Big(f, \frac{1}{n}, \mathcal{M}_{p,\lambda}(I_0, w)\Big) \le \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}, n \in \mathbb{N}$$

with a constant $C > 0$ independent of f and n.

Corollary 1. [2] Let $f \in \widetilde{\mathcal{M}}_{p,\lambda}(I_0)$, $1 < p < \infty$ and $0 \le \lambda < 1$. Then for every $r \in \mathbb{N}$ we have

$$\Omega^r\Big(f, \frac{1}{n}, \mathcal{M}_{p,\lambda}(I_0)\Big) \le \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0)}\Big\},\ n \in \mathbb{N}$$

with a constant $C > 0$ independent of f and n.

Corollary 2. [17] Let $f \in L_{p,w}(I_0)$, $1 < p < \infty$ and $w \in A_p(I_0)$. Then for every $r \in \mathbb{N}$ we have

$$\Omega^r\Big(f, \frac{1}{n}, L_{p,w}(I_0)\Big) \le \frac{C}{n^r}\Big\{E_0(f)_{L_{p,w}(I_0)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{L_{p,w}(I_0)}\Big\}, \quad n \in \mathbb{N}$$

with a constant $C > 0$ independent of f and n.

Similar result in Lebesgue spaces $L_p(I_0)$ was proved in [25, 26]. By using different modulus of smoothness, similar result was investigated in the paper [8] in weighted Lebesgue spaces $L_p(I_0, w)$, where $w \in A_p$.

The letters c, C are used for various constants, and may change from one occurrence to another.

2 Some auxiliary results

In this section we give some lemmas which we will need while proving our main results.

Lemma 1. Let $f \in \dot{W}^r \mathcal{M}_{p,\lambda}(I_0, w)$, $r \in \mathbb{N} \cup \{0\}$, $0 \le \lambda < 1$, $1 < p < \infty$ and $w \in A_p(I_0)$. Then for every $r \in \mathbb{N}$

$$\left\|\sigma_\delta^r(f)\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \le C\,\delta^r \left\|f^{(r)}\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)}$$

with a constant $C > 0$ independent of f.

Proof. Let $f \in \dot{W}^r \mathcal{M}_{p,\lambda}(I_0, w)$. Then $f^{(r)} \in \mathcal{M}_{p,\lambda}(I_0, w)$ and

$$\triangle_t^r f(x) = \int_0^t \cdots \int_0^t f^{(r)}(x + t_1 + t_2 + ... + t_r) dt_1 ... dt_r.$$

Then

$$\begin{aligned}
&\left\|\sigma_\delta^r(f)\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} = \left\|\frac{1}{\delta}\int_0^\delta |\triangle_t^r f(\cdot)| dt\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \left\|\frac{1}{\delta}\int_0^\delta \int_0^t \cdots \int_0^t |f^{(r)}(\cdot + t_1 + t_2 + ... + t_r)| dt_1 ... dt_r dt\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \delta^r \left\|\frac{1}{\delta^r}\int_0^\delta \cdots \int_0^\delta |f^{(r)}(\cdot + t_1 + t_2 + ... + t_r)| dt_1 ... dt_r\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \delta^r \left\|\frac{1}{\delta^{r-1}}\int_0^\delta \cdots \int_0^\delta \left\{\frac{1}{\delta}\int_{t_1+...+t_{r-1}}^{t_1+...+t_{r-1}+\delta} |f^{(r)}(\cdot + u)| du\right\} dt_1 ... dt_{r-1}\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \delta^r \frac{1}{\delta^{r-1}}\int_0^\delta \cdots \int_0^\delta \left\|\frac{1}{\delta}\int_0^{\delta+(r-1)\delta} |f^{(r)}(\cdot + u)| du\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} dt_1 ... dt_{r-1} \\
&= \delta^r \left\|\frac{1}{\delta}\int_0^{r\delta} |f^{(r)}(\cdot + u)| du\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le C\delta^r \left\|Mf^{(r)}\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le C\delta^r \left\|f^{(r)}\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)}.
\end{aligned}$$

Q.E.D.

Corollary 3. Let $f \in \dot{W}^r_{p,w}(I_0)$, $r \in \mathbb{N} \cup \{0\}$, $1 < p < \infty$ and $w \in A_p(I_0)$. Then for every $r \in \mathbb{N}$

$$\|\sigma_\delta^r(f)\|_{L_{p,w}(I_0)} \le C\delta^r \|f\|_{\dot{W}^{(r)}_{p,w}(I_0)}$$

with a constant $C > 0$ independent of f.

Corollary 4. [2] Let $f \in \dot{W}^r_{p,\lambda}(I_0)$, $r \in \mathbb{N} \cup \{0\}$, $1 < p < \infty$ and $0 \le \lambda < 1$. Then for every $r \in \mathbb{N}$

$$\|\sigma^r_\delta(f)\|_{\mathcal{M}_{p,\lambda}(I_0)} \le C\delta^r \ \|f\|_{\dot{W}^r_{p,\lambda}(I_0)}$$

with a constant $C > 0$ independent of f.

Corollary 5. Let $f \in \mathcal{M}_{p,\lambda}(I_0, w)$, $0 \le \lambda < 1$, $1 < p < \infty$ and $w \in A_p(I_0)$. Then for every $r \in \mathbb{N}$

$$\|\sigma^r_\delta(f)\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \le C\|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)}$$

with a constant $C > 0$ independent of f.

Proof. Using the triangle inequality we have

$$\begin{aligned}
\|\sigma^r_\delta(f)\|_{\mathcal{M}_{p,\lambda}(I_0,w)} &= \Big\|\frac{1}{\delta}\int_0^\delta |\triangle^r_t f(\cdot)|dt\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&= \Big\|\frac{1}{\delta}\int_0^\delta \Big|\sum_{s=0}^{r}\tbinom{r}{s}(-1)^{r+s+1}f(\cdot+st)\Big|dt\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \sum_{s=0}^{r}\tbinom{r}{s}\Big\|\frac{1}{\delta}\int_0^\delta |f(\cdot+st)|dt\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&= \Big\|\frac{1}{\delta}\int_0^\delta |f(\cdot)|dt\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{s=1}^{r}\tbinom{r}{s}\Big\|\frac{1}{\delta}\int_0^\delta |f(\cdot+st)|dt\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{s=1}^{r}\tbinom{r}{s}\Big\|\frac{1}{s\delta}\int_0^{s\delta} |f(\cdot+u)|du\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{s=1}^{r}\tbinom{r}{s}\Big\|\frac{1}{\delta}\int_0^{r\delta} |f(\cdot+u)|du\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + r2^r\Big\|\frac{1}{r\delta}\int_0^{r\delta} |f(\cdot+u)|du\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}.
\end{aligned}$$

Since the function f on $\mathbb{R}$ is 2π-periodic, without loss of generality, we can assume $r\delta < 2\pi$ and by boundedness of maximal operator in weighted Morrey spaces [16], we get

$$\begin{aligned}
\|\sigma^r_\delta(f)\|_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le \|f\|_{\mathcal{M}_{p(\cdot),\lambda(\cdot)}(I_0)} + r2^r C(p)\|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&= C(p,r)\|f\|_{\mathcal{M}_{p,\lambda}(I_0,w)}.
\end{aligned}$$

Q.E.D.

In the following lemma we give a characterization of $K-$functionals in terms of the modulus of smoothness in weighted Morrey spaces $\mathcal{M}_{p,\lambda}(I_0, w)$.

Lemma 2. Let $f \in \mathcal{M}_{p,\lambda}(I_0, w)$, $0 \le \lambda < 1$, $1 < p < \infty$ and $w \in A_p(I_0)$. Then for every $r \in \mathbb{N}$ and $0 < h \le c(r, \lambda, w)$ we have

$$c\ \Omega^r(f, h, \mathcal{M}_{p,\lambda}(I_0, w)) \le K_r(f, h)_{\mathcal{M}_{p,\lambda}(I_0,w)} \le C\ \Omega^r(f, h, \mathcal{M}_{p,\lambda}(I_0, w))$$

with constants $c, C > 0$.

Proof. Let $g \in W^r\mathcal{M}_{p,\lambda}(I_0,w)$. Then $g^{(r)} \in \mathcal{M}_{p,\lambda}(I_0,w)$ and hence $g^{(r)} \in L_1(I_0)$. Therefore we write

$$\triangle_t^r g(x) = \int_0^t \cdots \int_0^t g^{(r)}(x+t_1+t_2+...+t_r)dt_1...dt_r.$$

Using generalized Minkowski inequality and Theorem A, we have

$$\begin{aligned}
&\Omega^r(f,h,\mathcal{M}_{p,\lambda}(I_0,w)) := \sup_{|\delta|\le h} \Big\| \frac{1}{\delta}\int_0^\delta |\triangle_t^r g(x)| dt \Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \sup_{|\delta|\le h} \frac{1}{\delta}\int_0^\delta \Big\| \int_0^t \cdots \int_0^t \big|g^{(r)}(\cdot+t_1+t_2+...+t_r)\big| dt_1...dt_r \Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} dt \\
&\le h^r \Big\| \frac{1}{h^r}\int_0^h \cdots \int_0^h \big|g^{(r)}(\cdot+t_1+t_2+...+t_r)\big| dt_1...dt_r \Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&= h^r \| \frac{1}{h^{r-1}}\int_0^h \cdots \int_0^h \Big\{ \frac{1}{h}\int_{t_1+...+t_{r-1}}^{t_1+...+t_{r-1}+h} \big|g^{(r)}(\cdot+u)\big| du \Big\} dt_1...dt_{r-1} \|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le h^r \frac{1}{h^{r-1}}\int_0^h \cdots \int_0^h \Big\| \frac{1}{h}\int_0^{h+(r-1)h} \big|g^{(r)}(\cdot+u)\big| du \Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} dt_1...dt_{r-1} \\
&= h^r \Big\| \frac{1}{h}\int_0^{rh} \big|g^{(r)}(\cdot+u)\big| du \Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le c_r h^r \big\| M g^{(r)} \big\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \le c_r h^r \big\| g^{(r)} \big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}.
\end{aligned}$$

Hence, from the definition of $K_r(f,h)_{\mathcal{M}_{p,\lambda}(I_0,w)}$ we obtain

$$\begin{aligned}
\Omega^r(f,h,\mathcal{M}_{p,\lambda}(I_0,w)) &\le \Omega^r(f-g,h,\mathcal{M}_{p,\lambda}(I_0,w)) + \Omega^r(g,h,\mathcal{M}_{p,\lambda}(I_0,w)) \\
&\le c(\|f-g\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + h^r\|g^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}) \\
&\le cK_r(f,h)_{\mathcal{M}_{p,\lambda}(I_0,w)}
\end{aligned}$$

for any $f \in \mathcal{M}_{p,\lambda}(I_0,w)$.

In order to prove the converse inequality, we introduce a Steklov-type transform for $f \in \mathcal{M}_{p,\lambda}(I_0,w)$, $r \ge 1$, $h > 0$:

$$\begin{aligned}
&f_{r,h}(x) : \\
&= \frac{2}{h}\int_{\frac{h}{2}}^h \Big(\frac{1}{\delta^r}\int_0^\delta \cdots \int_0^\delta \sum_{s=0}^{r-1} \tbinom{r}{s} (-1)^{r+s+1} f(x+\frac{r-s}{r}(t_1+...+t_r)) dt_1...dt_r \Big) d\delta.
\end{aligned}$$

By simple calculations we have

$$\begin{aligned}&\left|f_{r,h}(x)-f(x)\right|\\&=\Big|\frac{2}{h}\int_{\frac{h}{2}}^{h}\Big(\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\sum_{s=0}^{r}\tbinom{r}{s}(-1)^{r+s+1}f(x+\frac{r-s}{r}(t_1+...+t_r))dt_1...dt_r\Big)d\delta\Big|\\&=\Big|\frac{2}{h}\int_{\frac{h}{2}}^{h}\Big(\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\sum_{s=0}^{r}\tbinom{r}{s}(-1)^{r+s+1}f(x+\frac{s}{r}(t_1+...+t_r))dt_1...dt_r\Big)d\delta\Big|\\&=\Big|\frac{2}{h}\int_{\frac{h}{2}}^{h}\Big(\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\triangle^r_{\frac{t_1+...+t_r}{r}}f(x)dt_1...dt_r\Big)d\delta\Big|.\end{aligned}$$

Taking the norm and applying generalized Minkowski inequality, we get

$$\begin{aligned}\left\|f_{r,h}-f\right\|_{\mathcal{M}_{p,\lambda}(I_0,w)}&\le\frac{2}{h}\int_{\frac{h}{2}}^{h}\Big\|\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\triangle^r_{\frac{t_1+...+t_r}{r}}f(x)dt_1...dt_r\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}d\delta\\&\le\sup_{\frac{h}{2}\le\delta\le h}\Big\|\frac{1}{\delta^r}\int_0^{\delta}\cdots\int_0^{\delta}\triangle^r_{\frac{t_1+...+t_r}{r}}f(x)dt_1...dt_r\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\&=\sup_{\frac{h}{2}\le\delta\le h}\Big\|\frac{1}{\delta^{r-1}}\int_0^{\delta}\cdots\int_0^{\delta}\Big(\frac{1}{\delta}\int_{t_1+...+t_{r-1}}^{\delta+t_1+...+t_{r-1}}\triangle^r_{\frac{t}{r}}f(x)dt\Big)dt_1...dt_{r-1}\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\&\le\sup_{\frac{h}{2}\le\delta\le h}\frac{1}{\delta^{r-1}}\int_0^{\delta}\cdots\int_0^{\delta}\Big\|\frac{1}{\delta}\int_0^{\delta+(r-1)\delta}|\triangle^r_{\frac{t}{r}}f(x)|dt\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}dt_1...dt_{r-1}\\&\le\sup_{\frac{h}{2}\le\delta\le h}\Big\|\frac{1}{\delta}\int_0^{r\delta}|\triangle^r_{\frac{t}{r}}f(x)|dt\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\&\le r\sup_{0\le\delta\le h}\Big\|\frac{1}{\delta}\int_0^{\delta}|\triangle^r_u f(x)|du\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}=r\ \Omega^r(f,h,\mathcal{M}_{p,\lambda}(I_0,w)).\end{aligned}$$

Differentiating $f_{r,h}$, we have

$$f^{(r)}_{r,h}(x)=\frac{2}{h}\int_{\frac{h}{2}}^{h}\frac{1}{\delta^r}\sum_{s=0}^{r-1}\tbinom{r}{s}(-1)^{r+s+1}\Big(\frac{r}{r-s}\Big)^r\triangle^r_{\frac{r-s}{r}\delta}f(x)d\delta.$$

Therefore,

$$\begin{aligned}|f^{(r)}_{r,h}(x)|&\le\frac{2}{h}\int_{\frac{h}{2}}^{h}\frac{1}{\delta^r}\sum_{s=0}^{r-1}\tbinom{r}{s}\Big(\frac{r}{r-s}\Big)^r|\triangle^r_{\frac{r-s}{r}\delta}f(x)|d\delta\\&=\frac{c_r}{h^r}\sum_{s=0}^{r-1}\tbinom{r}{s}\Big(\frac{r}{r-s}\Big)^r\frac{1}{h}\int_0^{h}|\triangle^r_{\frac{r-s}{r}\delta}f(x)|d\delta\\&=\frac{c_r}{h^r}\sum_{s=0}^{r-1}\tbinom{r}{s}\Big(\frac{r}{r-s}\Big)^r\frac{1}{\frac{r-s}{r}h}\int_0^{\frac{r-s}{r}h}|\triangle^r_u f(x)|du.\end{aligned}$$

Hence,

$$\begin{aligned}\|f_{r,h}^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le c_r h^{-r}\|\sigma^r_{\frac{r-s}{r}h} f\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\ &\le c_r h^{-r}\Omega^r(f,h,\mathcal{M}_{p,\lambda}(I_0,w)).\end{aligned}$$

Thus,

$$\begin{aligned}K_r(f,h)_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le \|f-f_{r,h}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + h^r\|f_{r,h}^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\ &\le C\,\Omega^r(f,h,\mathcal{M}_{p,\lambda}(I_0,w)).\end{aligned}$$

Q.E.D.

Let

$$f(x)\sim \frac{a_0}{2}+\sum_{k=1}^{\infty} a_k coskx + b_k sinkx \tag{2.1}$$

be the Fourier series of $f\in \widetilde{\mathcal{M}}_{p,\lambda}(I_0,w)$, $0\le\lambda\le 1$, $1<p<\infty$ and $S_n(x,f)$ be its n th partial sum. Under the condition $w\in A_1(I_0)$, using the method of proof of Lemma 1 and applying the appropriate results in weighted Lebesgue spaces given in [9], [10], we see that

$$\begin{aligned}\|f-S_n(\cdot,f)\|_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le CE_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)},\\ E_n(\tilde{f})_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le CE_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)},\end{aligned} \tag{2.2}$$

where $\tilde{f}$ is the conjugate function of f.

Lemma 3. Let $w\in A_p(I_0)$ and $r\ge 1$. Then for any $f\in \dot{W}^r\widetilde{\mathcal{M}}_{p,\lambda}(I_0,w)$, $0\le\lambda<1$, $1<p<\infty$, we have

$$E_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\le \frac{C}{n^r}\|f^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)},\quad n\in\mathbb{N}$$

with a constant $C=C(p,\lambda,w,r)$.

Proof. Let

$$f(x)\sim\sum_{k=0}^{\infty} a_k coskx + b_k sinkx$$

be the Fourier series of $f\in \widetilde{\mathcal{M}}_{p,\lambda}(I_0,w)$, $0\le\lambda<1$, $1<p<\infty$ and $S_n(x,f)$ be its n th partial sum. Then

$$\tilde{f}(x)\sim\sum_{k=0}^{\infty} b_k coskx - a_k sinkx.$$

Setting

$$A_k(x,f):=a_k coskx + b_k sinkx,\quad k\in\mathbb{N}$$

we have $f(x) = \sum\limits_{k=0}^{\infty} A_k(x,f)$ in the norm of $\mathcal{M}_{p,\lambda}(I_0,w)$. Since

$$\begin{aligned}
A_k(x,f) &= a_k coskx + b_k sinkx \\
&= a_k cos\Big(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\Big) + b_k sin\Big(kx + \frac{r\pi}{2} - \frac{r\pi}{2}\Big) \\
&= cos\frac{r\pi}{2}\Big[a_k cosk\Big(x + \frac{r\pi}{2k}\Big) + b_k sink\Big(x + \frac{r\pi}{2k}\Big)\Big] \\
&+ sin\frac{r\pi}{2}\Big[a_k sink\Big(x + \frac{r\pi}{2k}\Big) - b_k cosk\Big(x + \frac{r\pi}{2k}\Big)\Big] \\
&= A_k\Big(x + \frac{r\pi}{2k}, f\Big)cos\frac{r\pi}{2} + A_k\Big(x + \frac{r\pi}{2k}, \tilde{f}\Big)sin\frac{r\pi}{2}
\end{aligned}$$

and

$$A_k\Big(x, f^{(r)}\Big) = k^r A_k\Big(x + \frac{r\pi}{2k}, f\Big),$$

we get

$$\sum_{k=0}^{\infty} A_k\big(x,f\big) = A_0\big(x,f\big) + cos\frac{r\pi}{2}\sum_{k=1}^{\infty} A_k\Big(x + \frac{r\pi}{2k}, f\Big) + sin\frac{r\pi}{2}\sum_{k=1}^{\infty} A_k\Big(x + \frac{r\pi}{2k}, \tilde{f}\Big)$$

$$= A_0\big(x,f\big) + cos\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r}A_k\Big(x, f^{(r)}\Big) + sin\frac{r\pi}{2}\sum_{k=1}^{\infty}\frac{1}{k^r}A_k\Big(x, \tilde{f}^{(r)}\Big).$$

Then

$$\begin{aligned}
f(x) - S_n(x,f) &= \sum_{k=n+1}^{\infty} A_k(x,f) \\
&= cos\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\big(x, f^{(r)}\big) + sin\frac{r\pi}{2}\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\big(x, \tilde{f}^{(r)}\big).
\end{aligned}$$

Taking into account that

$$\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\big(x, f^{(r)}\big) = \sum_{k=n+1}^{\infty}\frac{1}{k^r}\Big[S_k\big(x, f^{(r)}\big) - S_{k-1}\big(x, f^{(r)}\big)\Big]$$

$$= \sum_{k=n+1}^{\infty}\frac{1}{k^r}\Big\{\Big[S_k\big(x, f^{(r)}\big) - f^{(r)}(x)\Big] - \Big[S_{k-1}\big(x, f^{(r)}\big) - f^{(r)}(x)\Big]\Big\}$$

$$= \sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)\Big[S_k\big(x, f^{(r)}\big) - f^{(r)}(x)\Big] - \frac{1}{(n+1)^r}\Big[S_n\big(x, f^{(r)}\big) - f^{(r)}(x)\Big]$$

and

$$\sum_{k=n+1}^{\infty}\frac{1}{k^r}A_k\big(x, \tilde{f}^{(r)}\big) = \sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r} - \frac{1}{(k+1)^r}\Big)\Big[S_k\big(x, \tilde{f}^{(r)}\big) - \tilde{f}^{(r)}(x)\Big]$$

$$-\frac{1}{(n+1)^r}\Big[S_n\big(x,\tilde{f}^{(r)}\big)-\tilde{f}^{(r)}(x)\Big],$$

by (2.2), we have

$$\begin{aligned}\|f-S_n(\cdot,f)\|_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le \sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)\Big\|S_k\big(\cdot,f^{(r)}\big)-f^{(r)}\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\ &+\frac{1}{(n+1)^r}\Big\|S_n\big(\cdot,f^{(r)}\big)-f^{(r)}\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\end{aligned}$$

$$\begin{aligned}&+\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)\Big\|S_k\big(\cdot,\tilde{f}^{(r)}\big)-\tilde{f}^{(r)}\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\ &+\frac{1}{(n+1)^r}\Big\|S_n\big(\cdot,\tilde{f}^{(r)}\big)-\tilde{f}^{(r)}\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\ &\le C\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)E_k\big(f^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}+\frac{1}{(n+1)^r}E_n\big(f^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}\\ &+C\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)E_k\big(\tilde{f}^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}+\frac{1}{(n+1)^r}E_n\big(\tilde{f}^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}.\end{aligned}$$

After simple calculations and using second relation of (2.2), we get

$$\begin{aligned}&\Big\|f-S_n(\cdot,f)\Big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\ &\le CE_n\big(f^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)+\frac{1}{(n+1)^r}\Big\}\\ &+CE_n\big(\tilde{f}^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\{\sum_{k=n+1}^{\infty}\Big(\frac{1}{k^r}-\frac{1}{(k+1)^r}\Big)+\frac{1}{(n+1)^r}\Big\}\\ &\le\frac{C}{(n+1)^r}E_n\big(f^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}.\end{aligned}$$

Hence,

$$\begin{aligned}E_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le \|f-S_n(\cdot,f)\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\\ &\le\frac{C}{n^r}E_n\big(f^{(r)}\big)_{\mathcal{M}_{p,\lambda}(I_0,w)}\le\frac{C}{n^r}\|f^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}.\end{aligned}$$

Q.E.D.

Now we will give the Bernstein inequality in Morrey spaces. Bernstein inequalities date back to 1912 when S.N. Bernstein proved the first inequality of this type for L_∞ norms of trigonometric polynomials. A generalization can be found in [5]; this result, which is credited to Zygmund, states that any trigonometric polynomial T of degree $n\in\mathbb{N}\cup\{0\}$ satisfies

$$\|T_n^{(k)}\|_{L_p(I_0)}\le Cn^k\|T_n\|_{L_p(I_0)}$$

for $1 < p < \infty$. Therefore we have the following:

Lemma 4. (Bernstein inequality in weighted Morrey spaces) Let $w \in A_p(I_0)$ and let $f \in \mathcal{M}_{p,\lambda}(I_0, w)$, $0 \le \lambda < 1$, $1 < p < \infty$. Then for every trigonometric polynomial T_n and $k \in \mathbb{N}$

$$\|T_n^{(k)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \le Cn^k \|T_n\|_{\mathcal{M}_{p,\lambda}(I_0,w)}, \quad n \in \mathbb{N} \cup \{0\}$$

with a constant C independent of n.

Proof. The proof is obtained similarly to that of Lemma 1 by using [17], where the Bernstein inequality was proved in $L_p(I_0, w)$. Q.E.D.

3 Proofs of main results

Proof of Theorem 1.3 Let $g \in \dot{W}^r \widetilde{\mathcal{M}}_{p,\lambda}(I_0, w)$. From Lemma 3 we get

$$\begin{aligned} E_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} &\le E_n(f-g)_{\mathcal{M}_{p,\lambda}(I_0,w)} + E_n(g)_{\mathcal{M}_{p,\lambda}(I_0,w)} \\ &\le \|f-g\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \frac{C}{n^r}\big\|g^{(r)}\big\|_{\mathcal{M}_{p,\lambda}(I_0,w)}. \end{aligned}$$

Since this inequality holds for every $g \in \dot{W}^r \widetilde{\mathcal{M}}_{p,\lambda}(I_0, w)$, by the definition of the $K-$functional and by Lemma 2, we get

$$E_n(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} \le C\, K_r\Big(f, \frac{1}{n}\Big)_{\mathcal{M}_{p,\lambda}(I_0,w)} \le C\, \Omega^r\Big(f, \frac{1}{n}, \mathcal{M}_{p,\lambda}(I_0, w)\Big).$$

Thus the proof is completed.

Proof of Theorem 1.4 Let $T_n \in \mathcal{P}_n$ be the polynomial of best approximation to f in $\widetilde{\mathcal{M}}_{p,\lambda}(I_0, w)$. For any integer $j = 1, 2, ...,$

$$\begin{aligned} K_r\Big(f, \frac{1}{n}\Big)_{\mathcal{M}_{p,\lambda}(I_0,w)} &= \inf_{g \in W^{(r)}\mathcal{M}_{p,\lambda}(I_0,w)} \Big\{\|f-g\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \frac{1}{n^r}\|g^{(r)}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\} \\ &\le \|f - T_{2^{j+1}}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \frac{1}{n^r}\|T^{(r)}_{2^{j+1}}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}. \end{aligned}$$

Using Lemma 4, we get

$$\|T^{(r)}_{2^{j+1}}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \le \|T^{(r)}_1 - T^{(r)}_0\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{i=0}^{j} \|T^{(r)}_{2^{i+1}} - T^{(r)}_{2^i}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}$$

$$\le C\Big\{\|T_1 - T_0\|_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{i=0}^{j} 2^{(i+1)r}\|T_{2^{i+1}} - T_{2^i}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}$$

$$\le C\Big\{E_1(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}$$

$$+ \sum_{i=0}^{j} 2^{(i+1)r}\Big\{E_{2^{i+1}}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + E_{2^i}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}\Big\}$$

$$\le C\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{i=0}^{j} 2^{(i+1)r}E_{2^i}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}$$

$$= C\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + 2^r E_1(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{i=1}^{j} 2^{(i+1)r}E_{2^i}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}.$$

Since

$$2^{(i+1)r}E_{2^i}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} \le 2^{2r} \sum_{m=2^{i-1}+1}^{2^i} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} \tag{3.1}$$

for $i \ge 1$, we have

$$\|T^{(r)}_{2^{j+1}}\|_{\mathcal{M}_{p,\lambda}(I_0,w)}$$

$$\le C\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + 2^r E_1(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + 2^{2r}\sum_{m=2}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}$$

$$\le C\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{m=1}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}.$$

Selecting j such that $2^j \le n < 2^{j+1}$, from (3.1) we get

$$E_{2^{j+1}}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} = \frac{2^{(j+1)r}E_{2^{j+1}}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}}{2^{(j+1)r}}$$

$$\le \frac{1}{n^r}2^{(j+1)r}E_{2^{j+1}}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} \le \frac{1}{n^r}\sum_{m=2^{j-1}+1}^{2j} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}.$$

Now by Lemma 2, we conclude that

$$\begin{aligned}
\Omega^r\Big(f,\frac{1}{n},\mathcal{M}_{p,\lambda}(I_0,w)\Big) &\le CK_r\Big(f,\frac{1}{n}\Big)_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le CE_{2^{j+1}}(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + \frac{1}{n^r}\|T_{2^{j+1}}\|_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&\le \frac{C}{n^r}\sum_{m=2^{j-1}+1}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} \\
&+ \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{m=1}^{2^j} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\} \\
&\le \frac{C}{n^r}\Big\{E_0(f)_{\mathcal{M}_{p,\lambda}(I_0,w)} + \sum_{m=1}^{n} m^{r-1}E_m(f)_{\mathcal{M}_{p,\lambda}(I_0,w)}\Big\}.
\end{aligned}$$

Thus the proof is completed.

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References

[1] S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctiones continues par des polynomes de degré donné,* Mem. Acad. Roy. Belgigue, 2 me serie, 4 (1912), 1–104.

[2] Z. Cakir, C. Aykol, D. Soylemez, A. Serbetci, *Approximation by trigonometric polynomials in Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., 39 (1) (2019), 1-17.

[3] F. Chiarenza, M. Frasca, *Morrey spaces and Hardy–Littlewood maximal function,* Rend. Math., **7** (1987), 273–279.

[4] R. Coifman, R. Rochberg, *Another characterization of BMO*, Proc. Amer. Math. Soc. 79 (1980), 249-254.

[5] R.A. Devore, G.G. Lorentz, *Constructive Approximation*, Springer-Verlag, Berlin, Heidelberg, New York, 1993.

[6] J. Duoandikoetxea, *Weights for maximal functions and singular integrals*, NCTH 2005 Summer School on Harmonic Analysis in Taiwan.

[7] L. Ephremidze, V.M. Kokilashvili, Y.E. Yildirir, *On the inverse inequalities for trigonometric polynomial approximations in weighted Lorentz spaces*, Proc. A. Razmadze Math. Inst. 144 (2007), 132-136.

[8] E.A. Haciyeva, *Investigation of the properties of functions with quasimonotone Fourier coefficients in generalized Nikolskii-Besov spaces*, Authors summary of candidate dissertation, Tbilisi, 1986 (in Russian).

[9] R. Hunt, B. Muckenhoupt, R. Wheeden, *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227-251.

[10] R.A. Hunt, W. Young, *A weighted norm inequality for Fourier series*, Bull. Amer. Math. Soc. 80 (1974), 274-277.

[11] I.I. Ibragimov, D.I. Mamedhanov, *A constructive characterization of a certain class of functions*, Dokl. Akad. Nauk SSSR, 223 (1975), 35-37, Soviet Math. Dokl., 4 (1976), 320-323.

[12] D.M. Israfilov and R. Akgun, *Approximation in weighted Smirnov-Orlicz classes*. J. Math. Kyoto Univ. 46 (2006), No. 4, 755-770.

[13] D.M. Israfilov, A. Guven, *Approximation by trigonometric polynomials in weighted Orlicz spaces*. Studia Math. 174 (2006), No. 2, 147-168.

[14] V.M. Kokilashvili, *Approximation on Smirnov-Orlicz classes*. Studia Math. 31 (1968), 43-59.

[15] V.M. Kokilashvili, Y. E. Yildirir, *On the approximation in weighted Lebesgue spaces*. Proc. A. Razmadze Math. Inst. 143 (2007), 103-113.

[16] Y. Komori, S. Shirai, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. 282 (2) (2009), 219-231.

[17] N.X. Ky, *On approximation by trigonometric polynomials in $L_{p,u}-$spaces*, Studia Sci. Math. Hungar., 28 (1993), 183-188.

[18] N.X. Ky, *Moduli of mean smoothness and approximation with A_p weights*, Ann. Univ. Sci., 40 (1997), 37-48.

[19] L. Lanzhe, *Sharp and weighted inequalities for multilinear integral operators*, RACSAM Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. 101 (2007), no. 1, 99-111.

[20] H.N. Mhaskar, *Introduction to the theory of weighted polynomial approximation*, Series in Approximation and Decompositions 7, World Scientific Publishing Co., Inc., River Edge, NJ 1996.

[21] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations.* Trans. Amer. Math. Soc., **43** (1938), 126–166.

[22] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207-226.

[23] E.S. Quade, *Trigonometric approximation in the mean,* Duke Math. J., 3(1937), No. 3, 529–543.

[24] S.B. Stechkin, *On the order of the best approximations of continuous functions*, Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, (1951), 219-242. (Russian)

[25] A.F. Timan, *Theory of Approximation of Functions of a Real Variable*, English translation 1963, Pergamon Press, The MacMillan Co., Russian original published in Moscow by Fizmatgiz 1960.

[26] A.F. Timan, M. F. Timan, *The generalized modulus of continuity and best mean approximation*, Dokl. Akad. Nauk. SSSR, 71 (1) (1950), 17-20.

[27] H. Triebel. *Hybrid Function Spaces, Heat and Navier-Stokes Equations*, Tracts in Mathematics 24, European Mathematical Society.

[28] A. Torchinsky, *Real Variable Methods in Harmonic Analysis*, Academic Press, San Diego, 1986.

[29] J.L. Walsh, H.C. Russel, *Integrated continuity conditions and degree of approximation by polynomials or by bounded analytic functions*, Trans. Amer. Math. Soc., 92 (1959), 355-370.